alone, the second, of order relations alone, the third, of order relations among the results of rational operations. This classification corresponds intimately to the characterizations of the theorems of algebra as falling into three analogous categories.

J. L. RILEY, *Secretary-Treasurer.*

CUSPIDAL ENVELOPE ROSETTES.

By WILLIAM F. RIGGE, Creighton University.

A point P moves in the line segment EG, Fig. 1, with simple harmonic motion of p cycles, while this segment makes q revolutions about A with uniform angular speed. Moritz[1] has exhaustively treated the case when the point A is in the line EG or in its prolongation. The writer[2] has shown that when the point A is out of the line EG and the rosette drawn is cuspidal, AL, the distance of EG from A, must be $n \sin \beta$ (in which $n = p/q$) and LR, the distance of R, the mid-point, or point of zero phase, of EG, from its point of tangency L on the *tangent* circle, must be $\cos \beta$. The point P remains on an ellipse whose conjugate semi-axis is unity ($= ER = RG$) and is always parallel to EG, whose major semi-axis $= n$, and whose center is the sine PR of the phase α distant from A, β being the eccentric angle of P.

When the point P is given a double rectilinear harmonic motion[3] with equal amplitudes but with periods in the ratio of m to 1, it may be conceived to move with simple harmonic motion of mn periods on the line segment DF, Fig. 2, while this line slides n times in a similar way along the line EG in one revolution of EG about A. For the sake of greater clearness these lines DF and EG are spaced a short distance apart in the figure. PS is then the sine of the phase of P on DF, while RS is the sine of the phase of S, the mid-point of DF, on EG. Hence the distance of the tracing point P from the *tangent* circle measured along the tangent line $GFED$ is

$$LP = -RL + RS + SP = -\cos \beta + \sin n\theta + \sin mn\theta,$$

in which θ is the phase of the circular motion about A. In the previous paper the point A was in the line EG. In the present A will be out of this line. The discussion will, as before, be confined to envelopes that are cuspidal.

Two Envelopes. As the points D and F are the limits of the excursions of P on this line segment DF, it is clear that these points themselves would trace the envelopes to all the lobes or loops or branches drawn by P, and that P must be on these envelopes when it is in phases 90° and 270°, respectively, on DF. The

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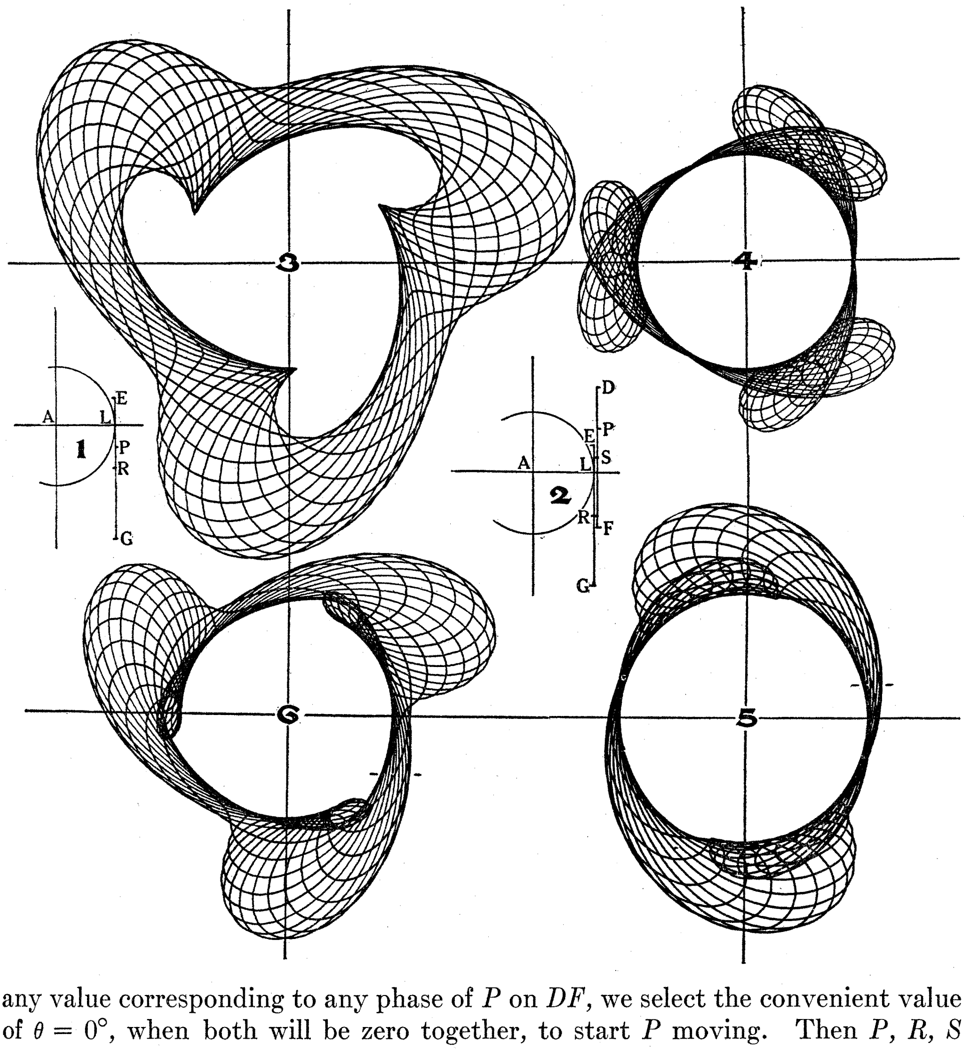
[1] "On the construction of certain curves in polar coördinates" by R. E. Moritz, in this MONTHLY, *1917*, 213–220.

[2] "Concerning a new method of tracing cardioids" by W. F. Rigge, in this MONTHLY, *1919*, 21–32. "Cuspidal rosettes," in this MONTHLY, *1919*, 332–340.

[3] "Envelope Rosettes," in this MONTHLY, *1920*, 151–157.

distance between the two envelopes measured in the line of motion of P is thus equal to 2, and the distance of P from them when in phase 0° or 180°, or multiples of them, must be -1 and $+1$.

The Starting Position of P for Cuspidal Envelopes. If D (or F) is to trace a cuspidal rosette, P must be set on a point of this rosette in phase 90° (or 270°) on DF, and then the phase of S on EG must also be set according to the nature of the curve. As the phase of S on EG, when $m-1$ is infinitesimal, may have



any value corresponding to any phase of P on DF, we select the convenient value of $\theta = 0°$, when both will be zero together, to start P moving. Then P, R, S are coincident in Fig. 2. But as the distance of P from the D (or F) envelope will then be $+1$ or -1, this starting position of P, R, S, in phase 0° must therefore be the distance unity on one side or other of the cuspidal rosette, that is, its starting ordinate $y_0 = -\cos\beta - 1$ or $-\cos\beta + 1$, while x_0 is always

$n \sin \beta$. From this it is evident that only one of the D and F envelopes can be cuspidal, except in a special case to be mentioned presently, when both may be so.

The Influence of β. When y_0 or $-\cos \beta \pm 1$ is small numerically, the D and F envelopes will as a rule intersect. Although unequal, they are always symmetrical on account of the positions of E and G on opposite sides of L. They are equal only when $y_0 = 0$, that is, when $\beta = 0°$ or $180°$, being coincident when n is odd, and symmetrically displaced when n is even. When y_0 or $-\cos \beta \pm 1$ is large numerically, the D and F envelopes cannot intersect, because E and G are then at very unequal distances from L and A. The inner one of the two will be cuspidal.

Illustrations. Fig. 3 with $n = 3$, $\beta = 30°$, $x_0 = n \sin \beta = 1.5$, $y_0 = -\cos \beta - 1 = -1.866$, shows the D or inner envelope cuspidal, while Fig. 4 with the same values of n, β, x_0, but with $y_0 = -\cos \beta + 1 = +0.134$, would seem at first sight to show two equal symmetrical and cuspidal envelopes. Only one however is in fact cuspidal and congruent to the inner envelope of Fig. 3. It is here the F envelope and is really smaller than the other or D envelope, because in this case $GL < EL$. The D envelope of Fig. 3 is in every way exactly equal to the F envelope of Fig. 4. When $\beta = 60°$ (but $n = 2$), as in Fig. 5, the inequality of the two envelopes is obvious, the smaller alone being cuspidal.

Transition Envelopes. Fig. 6 shows a transition envelope for $n = 3$. When the tracing point P is started in phases 0° with a smaller numerical value of y_0 than $-\cos \beta - 1$ as in Fig. 3, the cusp throws a lobe or a shoot like a growing bud, while the outer envelope is contracted. While y_0 is increasing in the positive direction, the two envelopes may become apparently equal or nearly so, although neither can be cuspidal. The cuspidal stage will be reached again when $y_0 = -\cos \beta + 1$ as in Fig. 4. After that the growth just mentioned will be reversed, until the value of $\beta' = 180° - \beta$ will again give cuspidal envelopes symmetrical however to those of β. For still larger positive values of y_0, the envelopes will tend to become nearly equal and circular.

Cuspidal envelope rosettes are best drawn when n is a small integer. When $n = 1$ one or both of the envelopes are cardioids for all values of β.

SEXAGESIMAL FRACTIONS AMONG THE BABYLONIANS.

BY FLORIAN CAJORI, University of California.

In view of the fact that certain writers have expressed the conviction[1] that, while the Babylonians operated with integers expressed in the sexagesimal system, they *did not use sexagesimal fractions*, it is worth while to refer briefly to a cuneiform tablet recently described by H. F. Lutz[2] of the University of Pennsylvania which unquestionably reveals the Babylonian use of sexagesimal frac-

[1] For example, see this MONTHLY, *1920*, 124.

[2] *American Journal of Semitic Languages and Literatures*, vol. 36, 1920, pp. 249–257. Tablet CBS 8536 in the University Museum, Philadelphia.